$$Q = -\frac{\beta\gamma\, d\log a + \gamma\alpha\, d\log b + \alpha\beta\, d\log c}{\alpha\beta + \alpha\gamma + \beta\gamma}.$$

Therefore by (3)

$$dA = \frac{\beta(d\log a - d\log c) + \gamma(d\log a - d\log b)}{\alpha\beta + \alpha\gamma + \beta\gamma},$$

$$dB = \frac{\gamma(d\log b - d\log a) + \alpha(d\log b - d\log c}{\alpha\beta + \alpha\gamma + \beta\gamma},$$

$$dC = \frac{\alpha(d\log c - d\log b) + \beta(d\log c - d\log a)}{\alpha\beta + \alpha\gamma + \beta\gamma}.$$

SOLUTION BY PROF. ASAPH HALL, NAVAL OBSERVAT'Y, WASH., D. C.

Projecting the sides b and c of a plane triangle on the side a we have

$$a = b\cos C + c\cos B;$$

and in a similar manner we find two more equations of this kind. Differentiating these, considering all the parts variable and noticing the condition

$$A + B + C = \pi = \text{constant},$$

we have the three symmetrical differential equations of a plane triangle,

$$da = \cos C.db + \cos B.dc + c\sin B.dA,$$
$$db = \cos A.dc + \cos C.da + a\sin C.dB,$$
$$dc = \cos B.da + \cos A.db + b\sin A.dC.$$

The quantities $\cos C.db$ and $\cos B.dc$ are the increments of the sides b and c projected on the side a; and the sum of these applied to da gives the total increment of the side a. Also $c\sin B$ is the perpendicular from the angle A on the opposite side, and the total increment of a divided by c sin B gives dA. For seconds of are we must multiply this ratio by 206264.8, the number of seconds in radius. Since

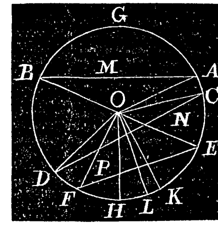
$$da = \frac{a.da}{a} = a.d\log a,$$

we may use the differentials themselves or $d\log a$, &c. One should notice the similarity of these differential equations to those of spherical trigonometry.

SOLUTION OF PART (2), PROB. 344, BY PROF. SEITZ (SEE P. 101).—Let P be the third random point, and EF the random chord, through it.

Draw the radius OL perpendicular to EF. Let p_0, p_1, p_2, p_3 be the respective probabilities that AB, CD, EF will intersect in 0, 1, 2, 3 points.

Let $EP = z$, $EF = z'$, $\angle EOL = \phi$, and $\angle KOL = \rho$. Then $z' = 2r \times \sin\phi$; an element of the circle at P is $r\sin\phi\, d\phi\, dz$, and for an elemental change in the direction of EF we have $d\rho$.



For non-intersection, the chord CD being between AB and EF, the limits of θ are 0 and π; of φ, 0 and θ; of ψ, 0 and φ; of μ, $\varphi-\theta$ and $\theta-\varphi$; and of ρ, $\psi-\varphi$ and $\varphi-\psi$. The result of these integrations must be multiplied by 6 to allow for the interchange of the chords.

For non-intersection, the chord EF having its extremities in the arc BD, the limits of θ are 0 and π; of φ, 0 and θ; of ψ, 0 and $\theta-\varphi$; of μ, $2\psi-\theta+\varphi$ $(=a)$ and $\theta-\varphi$; and of ρ, $\varphi+\psi$ and $\theta+\mu-\psi$ $(=\beta)$. This result must be multiplied by 4 to allow for the positions of EF, in which its extremities are in the arc AC, and for the positions of CD and EF in which their extremities are in the arc AGB.

The limits of ω are 0 and π; of x, 0 and x'; of y, 0 and y'; and of z, 0 and z'. Hence, since the whole number of ways the three chords can be drawn is $(\pi r^2)^3.\pi^3=\pi^6r^6$, we have

$$p_0=\frac{6}{\pi^6r^6}\int_0^\pi\int_0^\theta\int_0^\phi\int_{\phi-\theta}^{\theta-\phi}\int_{\psi-\phi}^{\phi-\psi}\int_0^\pi\int_0^{x'}\int_0^{y'}\int_0^{z'}r^3\sin\theta\sin\varphi\sin\psi\,d\theta d\varphi d\psi d\mu \times d\rho d\omega dx dy dz$$
$$+\frac{4}{\pi^6r^6}\int_0^\pi\int_0^\theta\int_0^{\theta-\phi}\int_a^{\theta-\phi}\int_{\phi+\psi}^{\beta}\int_0^\pi\int_0^{x'}\int_0^{y'}\int_0^{z'}r^3\sin\theta\sin\varphi\sin\psi\,d\theta d\varphi d\psi \times d\mu d\rho d\omega dx dy dz$$
$$=\frac{1}{3}-\frac{5}{\pi^2}+\frac{105}{4\pi^4}.$$

We will next find the probability that the chords CD and EF will both intersect AB. By reference to the first part of the solution we readily see that the probability that a third chord will also intersect AB is

$$P=\frac{16}{\pi^5}\int_0^{\frac{1}{2}\pi}\left\{2\int_0^\theta\int_{\theta-\phi}^{\theta+\phi}\sin^2\varphi d\varphi d\mu+2\int_\theta^{\frac{1}{2}\pi}\int_{\phi-\theta}^{\phi+\theta}\sin^2\varphi d\varphi d\mu\right\}^2\sin^2\theta\,d\theta$$
$$=\frac{2}{15}+\frac{1}{\pi^2}+\frac{49}{4\pi^4}.$$

Now P is evidently equal to the probability that the chords CD and EF will both intersect AB, and not each other, plus p_3; $\therefore \frac{1}{3}p_2+p_3=P$. (1)

The probability p, that AB and CD will intersect, is equal to the probability that they will intersect each other, and not EF, plus the probability that CD and EF will both intersect AB, and not each other, plus the prob. that AB and EF will both intersect CD, and not each other, plus p_3; hence

$$\tfrac{1}{3}p_1+\tfrac{2}{3}p_2+p_3=p. \qquad (2)$$

We also have
$$p_1+p_2+p_3=1-p_0, \qquad (3)$$

From (1), (2) and (3), knowing the values of P, p and p_0, we find

$$p_1=\frac{2}{5}+\frac{3}{\pi^2}-\frac{42}{\pi^4},\qquad p_2=\frac{1}{5}+\frac{3}{2\pi^2}+\frac{21}{4\pi^4},\qquad p_3=\frac{1}{15}+\frac{1}{2\pi^2}+\frac{21}{2\pi^4}.$$